A simple outline of spacetime thermodynamics*

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January 31, 2023

Abstract

By equating three temperature definitions that should simultaneously hold for an equilibrium thermodynamics of spacetime's degrees of freedom, it is immediately found that the latter only applies to spacetime surfaces with constant Newtonian gravitational potential, a result that was only rigorously proven in 2018. The laws of thermodynamics can correspondingly be rephrased in terms of spacetime's state variables at these surfaces.

Since the development of black hole thermodynamics by Bardeen et al. [1] in the 1970's, it has become clear that the laws of thermodynamics should not be limited to black hole horizons, but apply more broadly to the dynamics of spacetime as described by general relativity theory. Indications in this direction have come from, amongst others, the Tolman-Ehrenfest effect [2], the Fulling-Davies-Unruh effect [3], Jacobson's equation of state [4], and entropic gravity approaches. Additionally, for theories of (quantum) gravity involving a minimal length scale, some thermodynamic description for the microscopic constituents of spacetime simply must exist [5].

*Essay written for the Gravity Research Foundation 2023 Awards for Essays on Gravitation.

The laws of thermodynamics outline the constraints that systems obey in their statistical behaviour, usually in terms of state variables and for (quasi) equilibrium states. They can be summarized as (0) transitivity of thermal equilibrium; (1) energy conservation; (2) entropy increase with time; and (3) non-zero temperature. A thermodynamics of (curved) spacetime is thus obtained by defining an energy E, temperature T, and entropy S for the spacetime constituents, corresponding to the degrees of freedom N, that are in agreement with common thermodynamic concepts and compliant with the above laws, under the assumption of (local) equilibrium.

In order to uniquely determine the thermodynamic state variables E, T, S, and N in a dynamic spacetime context, one would need a system of at least four independent equations relating these quantities to each other as well as to general relativity theory. One of these equations has to be the connection between temperature and entropy as conjugate thermodynamic variables. Or, the conjugate temperature T_c is defined as a system's energy change with entropy:

$$T_c = \frac{\partial E}{\partial S} \tag{1}$$

The connection with spacetime dynamics can be made through the Davies-Unruh temperature, being the effective temperature T_a that is experienced by an uniformly accelerating observer in a quantized vacuum field [3]:

$$T_a = \frac{\hbar a}{2\pi c k_B} \tag{2}$$

with a the acceleration, and where $\hbar$ is the reduced Planck constant, c the speed of light in vacuum, and k_B the Boltzmann constant. If, through the equivalence principle, this acceleration is due to a gravitational field from a distant mass M (i.e., spacetime curvature induced by M), then a connection between Eqs. (1) and (2) can be made using Einstein's mass-energy relation:

$$E = Mc^2 = N_m m_1 c^2 \tag{3}$$

where the latter equation additionally introduces the system's degrees of freedom. It expresses that M is characterized by N_m constituents, each representing a unit-mass m_1 (which is to be determined later on).

This brings us to three equations for four unknowns. One can however proceed by explicitly imposing an *equilibrium* thermodynamics, at the small price of not knowing beforehand to which degrees of freedom the equilibrium applies. For systems in thermal equilibrium, the energy is equally distributed over the equilibrium degrees of freedom N_e that, from Eq. (1), have to be proportional to S. Correspondingly, the equipartition temperature T_e provides a measure for the energy per degree of freedom, given by:

$$T_e = \frac{2E}{k_B N_e} \tag{4}$$

By equating the above temperature expressions, $T_c \equiv T_a \equiv T_e$, one can hence identify whether or when (under what conditions) an equilibrium thermodynamics of spacetime is valid. This validity is expressed in terms of constraints on the degrees of freedom, in order to answer our key question: Which spacetime constituents, if any, would obey Eqs. (1) to (4) simultaneously? The answer is quite straightforward. Connecting Eqs. (2) and (4) for a gravitational pull g in presumed thermodynamic equilibrium yields:

$$g = \frac{4\pi GM}{N_e l_P^2} \tag{5}$$

given that $l_P^2 = \hbar G/c^3$ defines the Planck length l_P. This expression exactly corresponds to the Newtonian gravitational pull $g_N = 4\pi GM/A$ at any spherical surface A that is centered at the mass centre of M, if A represents N_e degrees of freedom of two-dimensional size l_P^2. Taking that Eq. (5) also holds for the Schwarzschild horizon, and by insertion of Eq. (3), additionally learns that in thermal equilibrium:

$$N_e = 4\pi N_m^2 \tag{6}$$

for a unit mass that equals half the Planck mass: $m_1 = m_P/2$, with $m_P = (c^2/G)l_P$ [6].

Most importantly, however, is that Eqs. (5) and (6) hold for all spacetime volumes with constant Newtonian gravitational potential, corresponding to all spherical surfaces A centered at the mass centre of M, where the gravitational acceleration and the surface normal are parallel. Such surfaces indeed have $N_m \propto R$ (cf. the Schwarzschild or de Sitter solution to Einstein's field equations) and $N_e \propto R^2$ in agreement with Eq. (6). Remarkably, this result was only rigorously proven in 2018 [7], although it had been suggested before [8, 9]. As noted in these references, this has its consequences for entropic views on gravity that do not hold without equipartition and hence are valid for spherically-symmetric equilibrium states only. The proportionality between the energetic degrees of freedom N_m and the radius R of A moreover indicates that the thermodynamics of the surface rather behaves like that of a one-dimensional quantum system [10].

By insertion of the surface degrees of freedom $N_A = N_e = 4\pi N_m^2$ into Eq. (4), one obtains:

$$T_A = \frac{\hbar c^3}{8\pi k_B G M} \tag{7}$$

as the surface temperature T_A for each concentric spherical surface around M. This result expresses a generalization of the Hawking temperature of black hole surfaces to all concentric spherical surfaces enclosing an isolated mass [11]. The temperature of curved spacetime as such represents a distance-dependent mass-reciprocal: If M denotes the Komar mass including the norm of a timelike Killing vector field $\sqrt{g_{tt}}$ [12], then $T_A\sqrt{g_{tt}}$ is constant for all spherical surfaces enclosing M, in agreement with the Tolman-Ehrenfest effect [2, 13]. Correspondingly, T_A also expresses the rate of the local gravitationally-delayed time with respect to proper time, and thus allows introducing the concept of thermal time [14].

Finally, equating Eq. (1) with Eq. (7) yields:

$$\partial S = 2\pi k_B N_m \partial N_m \tag{8}$$

or by integration on the surface A, up to a constant that is set to zero:

$$S_A = \pi k_B N_m^2 = \frac{k_B N_A}{4} \tag{9}$$

as an expression for the equilibrium spacetime (constituent) entropy of any mass-centered spherical surface, which is proportional to N_e as anticipated. Combining the above results, the laws of equilibrium thermodynamics can be given a proper interpretation for the Planck-scale constituents of (curved) spacetime, each representing a mass $m_P/2$:

(0) The temperature of spacetime is constant only on mass-centered spherical surfaces. Stated differently, non-spherically-symmetric mass distributions are not in thermal equilibrium (yet).

(1) Spacetime constituent numbers are conserved. Energy changes result from changes in surface constituent numbers (here not taking into account changes in angular momentum and electric charge): $\partial E = (k_B T_A/4)\partial N_A$ from Eqs. (1) and (9).

(2) The number of surface constituents cannot decrease: $\partial N_A/\partial t \geq 0$. This inequality expresses that gravity is an aggregating force. Stated differently, spacetime constituents cannot induce anti-gravitational effects (in contrast with e.g. Hawking radiation).

(3) The surface temperature T_A cannot vanish: $T_A > 0$. Given Eq. (7), this statement covers several considerations. First, one cannot have spacetime without gravity, or, spacetime has energy. Second, the degrees of freedom N_m must be finite within any finite radius, as the constituents of spacetime have a finite spatial extent l_P. Finally, any measured (Komar or other) mass must be finite too, also reflecting that masses cannot be accelerated up to or beyond the speed of light.

The above obviously reduces to black hole thermodynamics for $N_m = R/l_P$ and $N_A = 4\pi R^2/l_P^2$ [1], a limiting case that is valid for all causal horizons though. In their more general form, the laws of spacetime thermodynamics elaborate on Jacobson's expression of the Einstein field equations as an equation of state [4]. One has to conclude that, through these field equations, the dynamics of spacetime is the only one capable of maintaining temperature gradients in thermal equilibrium states without violating the laws of thermodynamics [15], but it can only do so in spherically-symmetric scenarios.

References

[1] J. M. Bardeen, B. Carter, and S. W. Hawking. The four laws of black hole mechanics. *Communications on Mathematical Physics*, 31(2):161–170, 1973.

[2] R. C. Tolman. On the weight of heat and thermal equilibrium in general relativity. *Physical Review*, 35:904–924, 1930.

[3] W. G. Unruh. Notes on black-hole evaporation. *Physical Review D*, 14(4):870–892, 1976.

[4] T. A. Jacobson. Thermodynamics of spacetime: The Einstein equation of state. *Physical Review Letters*, 75(7):1260–1263, 1995.

[5] T. Padmanabhan. Gravity is thermodynamics. *Current Science*, 109(12):2236, 2015.

[6] A. Keppens. What constitutes emergent quantum reality? A complex system exploration from entropic gravity and the universal constants. *Entropy*, 20(5):335, 2018.

[7] Z.-W. Wang and S. L. Braunstein. Surfaces away from horizons are not thermodynamic. *Nature Communications*, 9(2977), 2018.

[8] M. Visser. Conservative entropic forces. *Journal of High Energy Physics*, 2011(10), 2011.

[9] S. M. Carroll and G. N. Remmen. What is the entropy in entropic gravity? *Physical Review D*, 93(124052), 2016.

[10] Y. Xiao. Microscopic derivation of the Bekenstein-Hawking entropy for Schwarzschild black holes. *Physical Review D*, 101:046020, 2020.

[11] S. W. Hawking. Black hole explosions? *Nature*, 248(5443):30–31, 1974.

[12] A. Komar. Positive-definite energy density and global consequences for general relativity. *Physical Review*, 129(4):1873–1876, 1963.

[13] W. Kim. The effective Tolman temperature in curved spacetimes. *International Journal of Modern Physics D*, 26(14):1730025, 2017.

[14] C. Rovelli and M. Smerlak. Thermal time and Tolman–Ehrenfest effect: "temperature as the speed of time". *Classical and Quantum Gravity*, 28(7):075007, 2011.

[15] J. Santiago and M. Visser. Tolman temperature gradients in a gravitational field. *European Journal of Physics*, 40(2):025604, 2019.